RECENT DEVELOPMENTS IN LOCAL DISCONTINUOUS GALERKIN METHODS FOR LINEAR ELASTICITY[1]

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The discontinuous Galerkin (DG) methods [1] have received widespread interest in many computational fluid dynamic applications because of their inherent robustness and many other computational advantages. These advantages over the traditional counterparts are that they (i) can preserve local conservation, (ii) can provide arbitrary high-order of accuracy, (iii) are highly parallelizable since elements are discontinuous and the mass matrix is block diagonal, (iv) are easily amenable to hp-adaptivity since elements need not be conforming, and (v) allow non-congruent finite element discretization, and meshes with dissimilar adjoint element types. These features also drawing attention to and gaining a lot of interest in solving pure elliptic problems [2]. To this end, in this paper we present recent developments of the local discontinuous Galerkin (LDG) method to linear elasticity problems and outline the pro/cons over traditional paradigms.

Let us consider a linearly elastic body of open bounded volume of $\Omega \in \Re^n$ with bounded surface $\partial\Omega$. The body is in static equilibrium under the action of body forces $\mathbf{f} : \Omega \rightarrow \Re^n$, prescribed surface tractions $\bar{\mathbf{t}} : \Gamma_N \rightarrow \Re^n$ and prescribed displacement $\bar{\mathbf{u}} : \Gamma_D \rightarrow \Re^n$ where $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. The following second-order elasticity boundary value problem is considered. $\sigma_{ij,j} = f_i$ in Ω with $u_i = \overline{u}_i$ on Γ_D and $\sigma_{ij} n_j = \bar{t}_i$ on Γ_N; $\forall i = 1, \ldots, n$ where $\mathbf{n} = n_j$ is the unit outward normal to Γ_N, the stress tensor $\sigma = \sigma_{ij}$, and the strain tensor $\epsilon = \epsilon_{ij}$ are defined by $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$; $\epsilon_{ij} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$ in Ω. Let us consider the partition of the domain Ω into elements K such that $\Omega = \cup_i K_i$. The LDG method for linear elasticity can be stated as

$$\int_K \sigma^h_{ij} w_{,i} - \int_{\partial\Omega} \hat{\sigma}^h_{i,j} n_i w = \int_K f_i w \,\forall w \in U_k \tag{1}$$

$$\int_K \sigma^h_{ij} v = \int_K C_{ijkl}\epsilon^h_{kl} v; \quad \int_K \epsilon^h_{kl} v = -\frac{1}{2}\int_K (u^h_k v_{,l} + u^h_l v_{,k}) + \frac{1}{2}\int_{\partial K} (\hat{u}^h_k n_l + \hat{u}^h_l n_k) v \quad \forall v \in V_k \tag{2}$$

where $\hat{\sigma}^h_{ij} \equiv <\sigma^h_{ij}> - \widehat{A}_{ijkl}[\![n_k u^h_l]\!] - \widehat{B}_{ijk}[\![n_l \sigma^h_{lk}]\!]$, $\hat{u}^h_j \equiv <u^h_j> - \widehat{C}_{jk}[\![n_l \sigma^h_{lk}]\!] - \widehat{D}_{jkl}[\![n_k u^h_l]\!]$, $<(.)^h> \equiv ((.)^{h+} + (.)^{h-})/2$ is the average, and $[\![n_i (.)^h_j]\!] \equiv n^+_i (.)^{h+}_j + n^-_i (.)^{h-}_j$ is the jump operator. It can be shown that the above method is stable if and only if $\hat{A}_{ijkl} > 0$, $\hat{B}_{ijk} = -\hat{D}_{kij}$ in conjunction with the $\hat{u}^h_j = \overline{u}_i$ on Γ_D and $\hat{\sigma}^h_{ij} = \bar{t}_i$ on Γ_N. In addition, it can be shown that there exists a unique solution for these set conditions. If $\hat{C}_{jk} = 0$ then the above method results in a block diagonal mass matrix which allows for an easy computation of σ^h_{ij}. The phenomenon of gradient *super-convergence*, which include acceleration of convergence with increase in the number of Gauss points for the evaluation of $\int f_i w$ is demonstrated.

References

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