# REMARKS ON IRREGULAR HEXAHEDRAL FINITE ELEMENTS

**R. Falk**[a] and **P. Monk**[b]

[a]Department of Mathematics – Hill Center
110 Frelinghuysen Rd.
Rutgers University
Piscataway, New Jersey 08854-8019
falk@math.rutgers.edu

[b]Department of Mathematical Sciences
513 Ewing Hall
University of Delaware
Newark, Delaware 19716
monk@math.udel.edu

This paper is concerned with the approximation properties of several types of finite element spaces defined on irregular hexahedral elements obtained by trilinear mappings from a reference cube. These are used to approximate not only scalar functions in three dimensions, but also functions in $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$, i.e., 3-dimensional vector functions in $L^2$ whose divergence or curl is also in $L^2$. Such spaces arise naturally in many applications including the approximation of Maxwell's equations and the use of mixed and least squares finite element methods for second order elliptic equations. Although there has been a fairly extensive study of tetrahedral finite elements, most finite elements defined on cubes have not been studied to determine if they maintain key approximation properties when mapped to general hexahedrons, despite the fact that this is implicitly assumed since meshes of regular hexahedrons are quite restrictive in their use.

Because of the interelement continuity required (i.e., of the normal component of a vector function for finite element subspaces of $H(\mathrm{div}, \Omega)$ and of the tangential components of a vector function for finite element subspaces of $H(\mathrm{curl}, \Omega)$), the spaces on the physical element are defined by using special mappings (e.g., the Piola transform) of a set of reference functions defined on the unit cube. Of particular interest is the issue of what functions are necessary on the reference element to guarantee optimal order approximation by finite element subspaces of $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$. Such results were previously obtained for the space $H(\mathrm{div}, \Omega)$ in two dimensions, but in three dimensions the situation is considerably more complicated. For example, a general trilinear mapping of the unit cube produces a solid that will not, in general, have planar faces, but rather faces which can be hyperboloids.

Determining the minimum complexity of finite element spaces of $H(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$ needed to guarantee optimal order approximation is important when considering the alternative of using discontinuous Galerkin methods. Removing the requirement of interelement continuity of the normal or tangential components of a vector function allows the possibility of defining much simpler finite element spaces directly on the physical elements, rather than mapping from the reference element using special transformations. This may more than compensate for the more complicated variational formulations used in discontinuous Galerkin methods.