Goal-oriented a posteriori error estimates for finite element approximations using fundamental solutions

Thomas Grätsch

Department of Civil Engineering
The University of Kassel
Kurt-Wolters-Str. 3, 34109 Kassel, Germany
graetsch@uni-kassel.de
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The present paper deals with a posteriori error estimates for local quantities of interest by employing fundamental solutions. Here we restrict our approach to linear differential equations where the technique can be applied to all problems where a fundamental solution is known. The extension to nonlinear problems by solving a series of linear problems is still in progress.
Our approach for upper a posteriori error bounds of local quantities of interest in a pointwise sense is based on an integral representation via Green's function. Let

$$\mathcal{L}u = p \quad \text{on } \Omega \tag{1}$$

be a linear bvp of order $2m$ and assume that some boundary conditions are defined on a sufficiently smooth boundary $\Gamma = \Gamma_N + \Gamma_D$. Then for any local quantity or its derivative up to an order $2m-1$ we have

$$Q(u) = \int_\Omega G(y,x)p(y)\, d\Omega_y + \int_{\Gamma_N} G(y,x)t(y)\, ds_y \tag{2}$$

where $Q(u)$ is the local quantity and $G(y,x)$ the corresponding Green's function. Next, we split the Green's function into an unknown regular part u_R and the fundamental solution g,

$$G(y,x) = u_R(y,x) + g(y,x). \tag{3}$$

To obtain an approximation of Green's function, the regular part is calculated with finite elements. Therefore an Ansatz space $V_h \subset V$ is introduced and the discrete problem is to find a solution of

$$a(u_{R,h}, v_h) = -(t_g, v_h) \quad \forall v_h \in V_h \subset V \tag{4}$$

where $a(.,.)$ denotes the corresponding bilinear form, $(.,.)$ is the usual L_2-scalar product and t_g are the tractions which belong to the fundamental solution. Let

$$Q(u_h) = \int_\Omega G_h(y,x)p(y)\, d\Omega_y + \int_{\Gamma_N} G_h(y,x)t(y)\, ds_y \tag{5}$$

denote the recovered quantity where G_h is an approximation of (3) by using $u_{R,h}$ instead of u_R. By subtracting (5) from (2) we obtain for the local error

$$Q(u-u_h) = \int_\Omega (G-G_h)p(y)\, d\Omega_y + \int_{\Gamma_N} (G-G_h)t(y)\, ds_y. \tag{6}$$

Since the exact Green's function and the approximation both contain the fundamental solution we have

$$Q(u-u_h) = \int_\Omega (u_R - u_{R,h})p(y)\, d\Omega_y + \int_{\Gamma_N} (u_R - u_{R,h})t(y)\, ds_y = a(u, u_R - u_{R,h}). \tag{7}$$

Applying the Galerkin-orthogonality and the Cauchy-Schwarz inequality yields

$$|Q(u-u_h)| \le ||u_R - u_{R,h}||_E ||u-u_h||_E \tag{8}$$

which means that the local error is bounded by the error in the energy norm of the primal problem weighted with the error in the energy norm of the regular problem. To this result we can apply standard error estimation techniques such as the local Neumann problem approach proposed by Oden and Ainsworth.