THREE DIMENSIONAL VIBRATION ANALYSIS OF A THICK-WALLED CIRCULAR TORUS

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The strain-displacement equations of three-dimensional elasticity are derived for toroidal coordinates using the dyadic properties of orthogonal coordinate systems. The cross-section of the shell is modeled using a nine node Lagrangian finite element that is derived and evaluated directly in the toroidal coordinate system. The transformation equations between Cartesian coordinates and toroidal (r,θ,Ω) coordinates that were used in this study were given by Zhang and Redekop [1] as

$$x=(R+r\cos\theta)\sin\Omega,\qquad y=(R+r\cos\theta)\cos\Omega,\qquad z=r\sin\theta \tag{1}$$

where R is the radius of the torus, r and θ are the radius and angular measure of the thick circular shell, respectively, and Ω defines a rotation about the z axis (center of the torus). The displacement vector in toroidal coordinates is written as

$$\bar{u}=u_r\bar{e}_r+u_\theta\bar{e}_\theta+u_\Omega\bar{e}_\Omega \tag{2}$$

and the components of the strain tensor are computed following Chou and Pagano [2] as

$$\bar{\bar{\Omega}}=(1/2)\left[\bar{\nabla}\bar{u}+\left(\bar{\nabla}\bar{u}\right)^T\right] \tag{3}$$

The three dimensional character of the vibration analysis is preserved by assuming a solution that satisfies the governing equations and eliminates the circumferential coordinate Ω from the analysis. Similar to [3], assume

$$u_r(r,\theta,\Omega,t)=U(r,\theta)\cos m\Omega\cos\Omega t,\qquad u_\theta(r,\theta,\Omega,t)=V(r,\theta)\cos m\Omega\cos\Omega t,$$

$$u_\Omega(r,\theta,\Omega,t)=W(r,\theta)\sin m\Omega\cos\Omega t \tag{4}$$

where Ω is the circular frequency and m is the circumferential wave number. It can be shown that when R is large the vibration analysis agrees with that for an infinite thick-walled cylinder [4]. Typical nondimensional frequency results are compared with [3] (where only solid cross-sections are studied) in Table 1. for a solid torus. When the circumferential wave number m is zero the r,θ modes are uncoupled from the torsional, Ω modes. When m=1 or greater, the mode shapes reflect that cross-sectional displacement are coupled. Results are computed for a variety of cross-section dimensions and material properties.

Table 1. Comparison of finite element frequencies $\Omega=\Omega a(\Omega/G)^{1/2}$ for R=1.5, n=1, Ω=0.3 with reference [3], a =1, the radius of the shell, Ω=1, the density and G=1, the shear modulus.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9	Ω_{10}	Ω_{11}	Ω_{12}
FE	1.155	1.186	2.144	2.337	2.495	2.754	3.049	3.109	3.478	3.513	3.736	3.788
[3]	1.155	1.186	2.143	2.334	2.492	2.723	3.048	3.108	3.470	3.503	3.728	3.772

References

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