“DECONVOLUTION” TYPE MODELS USED IN FINITE DIFFERENCES AS A WAY TO RECOVER ENERGY CONSERVING SCHEMES

L. Georges and G. Winckelmans

Université catholique de Louvain (UCL), Mechanical Engineering Departement
Bât. Stévin, Place du Levant 2, B-1348 Louvain-la-Neuve, Belgium
georges@term.ucl.ac.be and winckelmans@term.ucl.ac.be

We are interested in simulating high Reynolds number flows using DNS and LES. In numerical methods, the continuum field u is projected onto a discrete “world” $\tilde{u}$, with a finite number of points. This projection can lead to different types of errors. We focus our attention on the conservation errors (on momentum and energy), limiting the analysis to second order central finite differences.

The one-dimensional convection-diffusion equation is analyzed first (also called *Burgers*' equation). It can be written in the following forms:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{uu}{2}\right) = \nu\frac{\partial^2 u}{\partial x^2} + f(orcing) \tag{1}$$

This equation has a “turbulent” behavior similar to that of the 3-D incompressible Navier-Stokes equations. The non-linear term can be expressed in the *advective* and *divergence* forms. These forms are equivalent in the continuum formulation but different numerically. When discretized, the advective and divergence forms both conserve momentum, but they both give rise to energy conservation errors that are $\mathcal{O}(h^2)$. They can however be modified to recover conservation. More specifically, one can use the following “modified” numerical schemes:

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\delta}{\delta x}\left(\frac{\tilde{u}\tilde{u}}{2} + \frac{\beta h^2}{2}\frac{\delta \tilde{u}}{\delta x}\frac{\delta \tilde{u}}{\delta x}\right) = \nu\frac{\delta^2 \tilde{u}}{\delta x^2} + f \tag{2}$$

$$\frac{\partial \tilde{u}}{\partial t} + \left(\tilde{u} + \beta h^2\frac{\delta^2 \tilde{u}}{\delta x^2}\right)\frac{\delta \tilde{u}}{\delta x} = \nu\frac{\delta^2 \tilde{u}}{\delta x^2} + f \tag{3}$$

where δ means second order central differences. The added term has the same form as the *non-linear model* proposed in LES, to reconstruct the *subfilter-scale stress*. This term must here be viewed as a generic supplement to the discretized Burgers' equation, without any necessary link with the concept of deconvolution due to an explicit filter. With the modified divergence scheme, Eq.(2), the energy is conserved up to $\mathcal{O}(h^4)$ when $\beta = -1/6$. Notice that β is negative: the link to an equivalent deconvolution procedure is thus rather weak (as β is then positive). With the modified advective scheme, Eq.(3), the energy is conserved exactly when $\beta = 1/3$: the numerical stencil is then exactly the same as the *skew-symmetric* form, $\frac{1}{3}\tilde{u}\frac{\delta\tilde{u}}{\delta x} + \frac{2}{3}\frac{\delta}{\delta x}\left(\frac{\tilde{u}\tilde{u}}{2}\right)$, which is indeed known to conserve energy. In this case, there likely is a link to a deconvolution procedure, expressed in advective form.

Direct Numerical Simulations (DNS) of the Burger's equation have been carried. The turbulence was fed using a white-noise forcing. Results are compared to the spectral method solution (as reference). The time-averaged spectra are analyzed. As example, the results using the modified advective form are presented: a 2048 points DNS is resolved using a spectral method (solid), the classical advective form (dot) and the modified advective form with $\beta = 1/3$ (dash). As expected, the modified version performs significantly better. Extension to the 3-D Navier-Stokes equations and to higher order methods is ongoing work.