# POLLUTION FREE FINITE ELEMENT PROCESSES FOR HELMHOLTZ EQUATION FOR ANY WAVE NUMBER

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The initial value problem (IVP) described by the wave equation gives rise to boundary value problem (BVP) called Helmholtz equation when the non-homogeneity is periodic. In many cases, the Helmholtz equation is defined over an unbounded domain with Sommerfeld's radiation boundary condition imposed at infinity. Using Dirichelet-to-Neumann (DtN) map this BVP with infinite domain is transformed over a finite domain in which the integral operator corresponding to non-local boundary conditions is further replaced by using Robin boundary condition. This form of the Helmholtz equation with Robin boundary conditions has been used as a standard benchmark model problem, hence we consider it here in this paper.

$$-\Delta\varphi - \kappa^2\varphi = f \ \text{ in } \Omega \ ; \ \varphi = 0 \ \text{ on } \Gamma_1 \ ; \ \frac{\partial\varphi}{\partial n} - i\,\kappa\varphi = q \ \text{ on } \Gamma_2 \qquad (1)$$

Since the Robin boundary condition on $\Gamma_2$ is complex, the solution $\varphi$ of BVP is complex as well. Investigation of the numerical solutions of BVP (1) for increasing wave number $\kappa$ using finite element method has been a subject of keen interest and investigation. The Galerkin method/weak form with local approximations of class $C^0$ (mostly linear) are used almost exclusively in investigating numerical solutions of BVP (1). While pollution [1] free numerical solutions for 1-D case have been presented using unconventional problem dependent approaches, it has been concluded that pollution free numerical solutions of Helmholtz equation in two and three dimensions are not possible [1].

Due to non-self adjoint nature of differential operator, Galerkin method/Weak form is variationally inconsistent (VIC) [2]. 2) VIC integral forms yield degenerate computational processes in which spurious solutions are possible. 3) The variational consistency (VC) of the VIC integral forms resulting from Galerkin method/weak form for non-self adjoint differential operators can not be restored using means that are mathematically as well as physically justifiable. 4) We show that spuriousness of the numerical solutions for coarser discretization is a consequence of the VIC. 5) VC integral forms of the BVP (1) can be achieved by using least squares processes (LSP) utilizing the strong form of the governing differential equations. 6) The theoretical solution of (1) are undoubtedly of higher order global differentiability and the global differentiability of a numerical solution can only be increased by increasing the order ($k$) of the space and not by *h, p, hp* – adaptive processes. 7) If we wish the numerically computed solutions to possess the same continuity and differentiability characteristics as theoretical solutions of BVP (1), then the use of higher order spaces is essential. We show mathematically as well as computationally that VC integral forms resulting from LSP with higher order global differentiability of the local approximations provide a framework in which the computed solutions of Helmholtz equation (1) are naturally pollution free for any wave number. Such solutions when computed in progressively increasing order ($k$) space possess the same differentiability characteristics as the theoretical solution and in the limit $k \to \infty$, yield the true theoretical solutions.

**References**

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